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The Blaschke-Steinhardt Point of a Planar Convex Set

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In memory of Fritz Steinhardt who passed away on April 14, 1993

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Abstract—There is a broad class of geometric optimization problems in $\mathbb{R}^n$ associated with minimizing "mixed volumes" of polar bodies. In this note, we compute the point (of duality) of a planar convex figure that minimizes the length (perimeter) of its polar set. This point was first described by W. Blaschke and then later investigated by F. Steinhardt, W. Firey, E. Lutwak, and others. The algorithm is illustrated for a few examples.

Keywords—Convex geometry, Duality, Geometric analysis, Mixed volume, Polar optimization.

1. PRELIMINARIES

1.1. The Support Function

Let K be a planar convex domain such that the origin O is an interior point of K, and let S^1 denote the unit circle centered at O. If u is a unit vector in $\mathbb{R}^2$, i.e., $u \in S^1$, let $L(u)$ denote the line that is perpendicular to u such that it meets K but not the interior of K. The direction of u is determined as follows: u is to point into the halfplane determined by $L(u)$ which does *not* contain K. $L(u)$ is called the support line of K, and the distance between O and $L(u)$ is the support function of K which is usually denoted by $H(u)$. Since the form of $H(u)$ obviously depends on the domain K, the support function is sometimes written as $H(K,u)$. More formally, we have the following definition.

Definition. *Let K be a nonempty convex set in $\mathbb{R}^2$, and let S^1 denote the unit circle centered at the origin. The restricted support function $H : S^1 \longrightarrow \mathbb{R}$ of K is defined to be*

$$H(K,u) = \sup\{\langle u, x\rangle \mid x \in K\},$$

for $u \in S^1$ and where $\langle u, x\rangle = \sum u_i x_i$ is the inner product of u and x.

One important consequence of the support function is that there is an isomorphism of the space of compact convex sets onto the space of convex positive homogeneous functions in the plane. There is also an isomorphism onto the restriction of such functions to the boundary of the unit ball in $\mathbb{R}^2$ centered at the origin. Note that $u \in S^1$ is uniquely specified by $(\cos\theta, \sin\theta)$, where θ is the angle between u and the positive x-axis, and so the support function is also commonly denoted by $H(K,\theta)$.

1.2. The Gauge Function

The notion of a distance (or gauge) function was first introduced by Minkowski in the study of convex bodies. Three equivalent formulations of the gauge function are motivated.

DEFINITION (*Constructive*). *Let K be a nonempty convex set in $\mathbb{R}^2$, and let x be any point (within, on, or outside K). Select O from the interior of K. It is clear that the ray from O through x intersects the boundary ∂K in exactly one point. Call this point y. We define the distance function as follows:*

$$d : \mathbb{R}^2 \longrightarrow [0, \infty).$$

For $x \in \partial K$, $d(x) = 1$. For $x \notin \partial K$, the ray from O through x will intersect ∂K (in exactly one point) at y. Hence, there exists $\lambda > 0$ such that $x = \lambda y$. In this case, define

$$d(x) = \lambda.$$

Finally, for $x = O$ do the obvious and set $d(O) = 0$.

DEFINITION (*Conventional*). *The distance function is defined by*

$$d(x) = \inf \{\lambda \geq 0 \mid x \in \lambda K\},$$

where λK represents the image of K under a homothetic transformation in the ratio $\lambda : 1$. It is not difficult to show that d is a convex function [1].

DEFINITION (*Computational*). *To actually perform calculations, we need to choose a specific metric. Since $x = \lambda y$,*

$$d(x) = \lambda = \frac{d(O, x)}{d(O, y)},$$

where $d(O, x)$ is the Euclidean distance function from O to x; i.e.,

$$d(O, x) = \sqrt{\sum x_i^2}.$$

The geometry of K enters into the denominator term $d(O, y)$ since $y \in \partial K$, and hence, $d(x)$ is sometimes written as $d(K, x)$ to denote this dependence. Again, we can write the restricted distance function as $d(K, u)$, where u is restricted to the unit circle, or as $d(K, \theta)$. In the case of a circle $d(O, y)$ is constant (in all directions), and therefore, symmetric in O and y, and so the gauge function is equivalent to the "usual" distance function of Euclidean geometry. In general, however, the gauge of a convex body is a complicated function which is not symmetric in its arguments. See, for instance, [1–3] for further details.

1.3. The Radial Function

An important function closely related to the gauge function which provides a means to describe K is the radial function $r(K, x)$.

DEFINITION. *Let K be a planar convex domain such that the origin O is an interior point of K. The radial function r is defined by*

$$\begin{aligned} r(K, x) &= \max \{\lambda \geq 0 \mid \lambda x \in K\} \\ &= \frac{1}{d(K, x)}, \end{aligned}$$

where $x \in \mathbb{R}^2 \setminus \{O\}$. The restricted radial function $r(K, u)$ is defined for $u \in S^1$.

1.4. The Euclidean Duality

Observe that since the gauge function d is convex, it is the support function of some *other* compact convex set in the plane. This convex body is denoted by $(K^*;x)$, the dual (or polar reciprocal) body of K:

$$(K^*;x) = \left\{x' \in \mathbb{R}^2 \mid \langle x' - x,\, y - x\rangle \leq 1, \text{ for all } y \in K\right\}.$$

The polar body of K with respect to the unit circle centered at the origin O is written as K^*. For a convex body K that contains O in its interior, the Euclidean duality is as follows [2,4]:

> The support function $H(K,\bullet)$ of a convex body K is equal to the gauge function $d(K^*,\bullet)$ of the polar body K^*.

And vice-versa:

> The gauge function $d(K,\bullet)$ of a convex body K is equal to the support function $H(K^*,\bullet)$ of the polar body K^*.

For a more complete exposition on duality, see [5,6].

Using the Euclidean duality, the polar body K^* can also be described by

$$H(K^*,\theta) = \frac{1}{r(K,\theta)},$$

$$r(K^*,\theta) = \frac{1}{H(K,\theta)},$$

which expresses a simple way to find the boundary points of K^* from the supporting planes of K (see Section 3.2). Furthermore, since the area of K, $A(K)$, is given by

$$A(K) = \int_0^{2\pi}\int_0^{r(K,\theta)} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} r(K,\theta)^2\,d\theta,$$

it follows that the area of the polar set $(K^*;x_\circ)$ is given by

$$A(K^*;x_\circ) = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{H(K,\theta)^2}.$$

It is also well-known that the length (or perimeter) of K is

$$L(K) = \int_0^{2\pi} H(K,\theta)\,d\theta,$$

and so the length of $(K^*;x_\circ)$ is simply

$$L(K^*;x_\circ) = \int_0^{2\pi}\frac{d\theta}{r(K,\theta)}.$$

1.5. Interpreting Area and Length in Terms of Mixed Areas

A convenient and useful way to describe the notions of area and length is through "mixed areas." If K_1 and K_2 are plane convex figures, then the mixed area $A(K_1,K_2)$ is defined by

$$A(K_1,K_2) = \frac{1}{2}\int_0^{2\pi} H(K_1,\theta)\,ds(K_2,\theta),$$

where $ds(K_2,\theta)$ is the arclength measure of K_2 (see [7]). $A(K_1,K_2)$ is symmetric in its arguments [1] and

$$A(K_1,K_1) = A(K_1),$$

and for the unit circle S^1 centered at the origin

$$2A\left(K_1,S^1\right) = 2A\left(S^1,K_1\right) = L(K_1).$$

2. A FAMILY OF GEOMETRIC OPTIMIZATION PROBLEMS

There is a broad class of optimization problems in $\mathbb{R}^n$ associated with minimizing the "mixed volume" of polar bodies. Inherent numerical difficulties associated with volume computation in high dimensions, however, limit the investigation of such problems [8,9]. See also [10]. Furthermore, the somewhat isolated nature of the mixed volume problem in the geometric literature has compounded the neglect, and although the following problems are easily formulated in $\mathbb{R}^n$, computational considerations will restrict our implementation to the plane.

2.1. The Santaló Point

Given a convex polytope P_m in $\mathbb{R}^n$ described in terms of its vertex set $\{A_1, A_2, \dots, A_m\}$ and a point Q within the interior of P_m, P_m°, called the point of duality, there is associated with P_m and Q a dual (or polar reciprocal) figure $(P_q^*; Q) = \{A_1^*, A_2^*, \dots, A_q^*\}$. The dual figure is described analytically by

$$(P_q^*; Q) = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \le 1,\ y \in P_m\},$$

and geometrically polarity is defined through a 1-1 correspondence with the input polygon [11,12] (the r *facets* of P_m correspond to the *vertices* A_i^*, and so in the plane $m = r = q$, but in space and higher dimensions the values of m and r are generally different). The volume of $(P_q^*; Q)$ will be denoted by $V(P_q^*; Q)$ and is a function of P_m and the point of duality Q. An interesting optimization problem associated with the construction of the dual figure $(P_q^*; Q)$ is to locate Q within P_m such that the volume $V(P_q^*; Q)$ is minimized.

The point which minimizes the volume of $(K^*; x)$ is known as the Santaló point $S(K)$; i.e., $S(K)$ is the solution to the optimization problem

$$\min_{x \in K^\circ} V(K^*; x),$$

where

$$V(K^*; x) = \frac{1}{n}\int_{S^{n-1}} \frac{d\omega}{H(K,\omega)^n},$$

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, and $d\omega$ is the $(n-1)$-dimensional measure on S^{n-1}.

The Santaló point is unique and affine invariant [13], and several characterizations/definitions of $S(K)$ can be motivated [14,15] (see also [16–18]). An algorithm to determine $S(K)$ in the planar case was described in a previous note [19]. The product

$$V(K)V(K^*; S(K))$$

is an affine invariant of K and is bounded by functions of the dimension. For example, for any symmetric convex region K in $\mathbb{R}^2$,

$$8 \le V(K)V(K^*; S(K)) \le \pi^2,$$

with equality holding in the left inequality for rectangular regions and in the right inequality for circles. For general K,

$$V(K)V(K^*; S(K)) \ge \frac{27}{4}$$

with equality holding for triangular regions [20]. A widespread conjecture is that

$$V(K)V(K^*; S(K)) \ge \frac{4^n}{n!}$$

is true for symmetric convex bodies in any dimension n. The most complete results presently known about these bounds can be found in [21].

2.2. The Mixed Volume Optimization Problem

A more general class of optimization problem is to minimize the "mixed" volume

$$V_m(K)V_m((K^*;x))$$

for $1 < m \leq n$, where

$$V_m(K) = V\,(\underbrace{K,\ldots,K}_{m},\underbrace{S^{n-1},\ldots,S^{n-1}}_{n-m}),$$

is the so-called m^{th} (integral) cross-sectional measures [22]. Recall that the volume of a body K in a linear family of convex bodies

$$K = \lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_r K_r$$

is an n^{th} degree homogeneous polynomial in the parameters $\lambda_1, \lambda_2, \ldots, \lambda_r$:

$$V\,(K) = \sum_{\rho_1=1}^{r} \cdots \sum_{\rho_n=1}^{r} V_{\rho_1\rho_2\cdots\rho_n}\lambda_{\rho_1}\lambda_{\rho_2}\cdots\lambda_{\rho_n},$$

where the products of the λ_i which differ only in the order of the factors have the same numerical coefficients. The coefficients $V_{\rho_1\rho_2\cdots\rho_n}$ of $V\,(K)$ are the mixed volumes of $K_1, \ldots, K_r$, and are also sometimes written as $V\,(K_{\rho_1}, \ldots, K_{\rho_n})$. If the bodies $K_1, \ldots, K_r$ are all equal to the body K, then

$$V\,(K_1, \ldots, K_r) = V(K, \ldots, K) = V(K),$$

or in accord with the previous notation,

$$V\,(K) = V_n(K).$$

The $(n-1)$-dimensional boundary area $S'(K)$

$$S'(K) = nV_{n-1}(K)$$

is another particular case of this formulation. See [4] for a very beautiful and complete treatment of the subject; [23] offers a more concise survey.

2.3. The Blaschke-Steinhardt Point

A few results on the mixed volume problem are available. For instance, Firey [24] proved for $m = (n+1)/2$ and odd n, and Lutwak [25] for all $m \leq (n+1)/2$ that

$$V_m(K)V_m(K^*;x) \geq \left(V\,\left(S^{n-1}\right)\right)^2.$$

The Firey inequality for mean width is thus

$$V_1(K)V_1(K^*;x) \geq \left(V\,\left(S^{n-1}\right)\right)^2,$$

with equality holding for the ball with center at the origin. In the plane this becomes

$$L(K)L(K^*;x) \geq 4\pi^2,$$

where $L(K)$ denotes the length (or perimeter) of K. This result was already shown by Steinhardt [26], and has its historical origin in [27]. Related work on this problem can be found in [28–31]. For a fixed region K, $L(K)$ is easily determined and so the general optimization problem is to choose the point of duality so that $L(K^*;x)$ is minimum; i.e.,

$$\min_{x \in K^\circ} L(K^*;x),$$

where

$$L(K^*;x) = \int_0^{2\pi} \frac{d\theta}{r(K,\theta)}.$$

The solution to this optimization problem will be called the Blaschke-Steinhardt point and denoted by $B(K)$. An algorithm to determine $B(K)$ is now described.

3. MINIMIZING THE LENGTH OF THE POLAR POLYGON

NOTATION. Let $(P_n^*; Q)$ represent the polar polygon computed with respect to the duality point $Q = (\alpha, \beta)$, and let $A(P_n^*; Q)$ represent the area and $L(P_n^*; Q)$ the length (perimeter) of this convex set. We will also use the notation $L(P_n^*; x)$ for the polar length when the duality point x is constrained to vary on a circle. A circle centered at y with radius δ will be written as $C(y; \delta)$.

3.1. The Main Idea of the Algorithm

The main idea of the algorithm is as follows. Since $A(P_n^*; Q) \longrightarrow \infty$ as $Q \longrightarrow \partial P_n$, the perimeter $L(P_n^*; Q) \longrightarrow \infty$ and so it makes sense to select an initial Q which is "centrally" located and "easy" to compute. To reduce the number of "iterations" of the algorithm, it is desirable to choose the initial duality point as close to the final solution as possible. Since the center-of-gravity of P_n, $g(P_n)$, is well-known to be the "balance" point of P_n (see [32]), it is a natural first choice to consider. A circular search centered at $g(P_n)$ with radius δ_1 is thus performed, and the length $L(P_n^*; x)$ is computed for $x \in C(g(P_n); \delta_1)$ (say for every one degree over the circle). As long as δ_1 is not too large (a value $\delta = .01$ worked well in practice), and as long as $g(P_n)$ is not the solution point, the length $L(P_n^*; x) < L(P_n^*; g(P_n))$ for some $x \in C(g(P_n); \delta_1)$—call this location $x^*_{(i)}$; compute $L(P_n^*; x^*_{(i)})$, and then perform another circular search centered at $g(P_n)$ with a value $\delta_2 > \delta_1$. Continue with the iterations until the smallest length term from each circular search $L(P_n^*; x)$ fails to decrease. The last point obtained Q^* is a local optimal solution. A final search in a small neighborhood of Q^* will verify that the point obtained is a local optimal. Convexity conditions on the form of the functional would ensure that the local optimal is a global solution; it is not clear, however, that the functional $L(K^*; x)$ is convex from the *circular* search data. Other evidence (such as graphing the functional $L(K^*; x)$) indicates that $L(K^*; x)$ *is* convex and that the local solution is unique, and hence, is the global minimizer. A proof of the convexity of $L(K^*; x)$ is not available.

In summary, then, the search begins from the initial duality point $g(P_n)$, and the algorithm proceeds by computing the (length of the) dual polygons for points selected on a small circle centered at $g(P_n)$. One "iteration" of the algorithm is completed after the minimum length polar polygon is chosen from the list of its values on the circle. The circle is then enlarged and the search repeated until the minimal length no longer decreases. The algorithm terminates at a local optimal solution. The formal details of the algorithm follow.

3.2. The Planar Optimization Problem

THE PLANAR OPTIMIZATION PROBLEM: $\min_{Q \in P_n^\circ} L(P_n^*; Q)$.

The input data of this problem are the vertices $A_i = (a_i, b_i)$, $i = 1, \dots, n$ of the convex polygon P_n and the initial point of duality $Q = (\alpha, \beta)$. The procedure to construct $(P_n^*; Q)$ is a two step process.

Input: $P_n = \{ A_1, A_2, \dots, A_n \}$; $A_i = (a_i, b_i)$ $i = 1, \dots, n$; $Q = (\alpha, \beta)$.

Output: $(P_n^*; Q) = \{ A_1^*, A_2^*, \dots, A_n^* \}$; $A_i^* = (a_i^*, b_i^*)$ $i = 1, \dots, n$.

(i) Determine $L_i : y = m_i x + c_i$.

(ii) Determine $A_i^* = \left(\dfrac{-m_i}{c_i + m_i\alpha - \beta} + \alpha, \dfrac{1}{c_i + m_i\alpha - \beta} + \beta \right)$.

The polar polygon $(P_n^*; Q)$ is the convex hull of the polar vertices A_i^*, $i = 1, \dots, n$. To solve the minimization problem we adopt the following search for a given $\epsilon > 0$ and $\delta_i > 0$.

Step 0. Set $i = 1$.

Step 1. Determine the initial point of duality $g(P_n) \in P_n^\circ$ and construct $(P_n^*; g(P_n))$.

Step 2. Perform a circular search over $C(g(P_n); \delta_i)$; i.e., for a uniform selection of points $x \in C(g(P_n); \delta_i)$, construct $(P_n^*; x)$, compute the length $L(P_n^*; x)$, and then choose the smallest value from this list:

$$x_{(i)}^* = \min_{x \in C} L\left(P_n^*; x\right).$$

(i) If $L(P_n^*; g(P_n)) \leq L(P_n^*; x_{(i)}^*)$, then stop. Repeat the search with $\delta_i/10$.

(ii) If $L(P_n^*; g(P_n)) > L(P_n^*; x_{(i)}^*)$, then continue.

Step 3. Perform a circular search over $C(g(P_n); \delta_{i+1})$, where $\delta_{i+1} = \delta_i + \delta_{i-1}$ (with $\delta_0 = \delta_1$). Corresponding to the minimum polar length is the point of duality $x_{(i+1)}^*$.

(i) If $L(P_n^*; x_{(i)}^*) \leq L(P_n^*; x_{(i+1)}^*)$, then stop. $x_{(i)}^*$ is the optimal solution.

(ii) If $L(P_n^*; x_{(i)}^*) > L(P_n^*; x_{(i+1)}^*)$, then repeat the search with $\delta_{i+2} = \delta_{i+1} + \delta_i$.

Step 4. Continue the search until termination. Then $x_{(i)}^* \longrightarrow Q^* = B(K)$.

There are many variations on the general search that can be implemented, and we choose this approach mainly for the simplicity of implementation.

3.3. Computational Experience

Standard examples are depicted in the three, four, six, and ten vertex case in Figures 1–4. Table 1 illustrates the convergence of the search algorithm for Figure 1, and Table 2 summarizes the relevant information from Figures 1–4: the initial duality point $g(P_n)$ and length of the

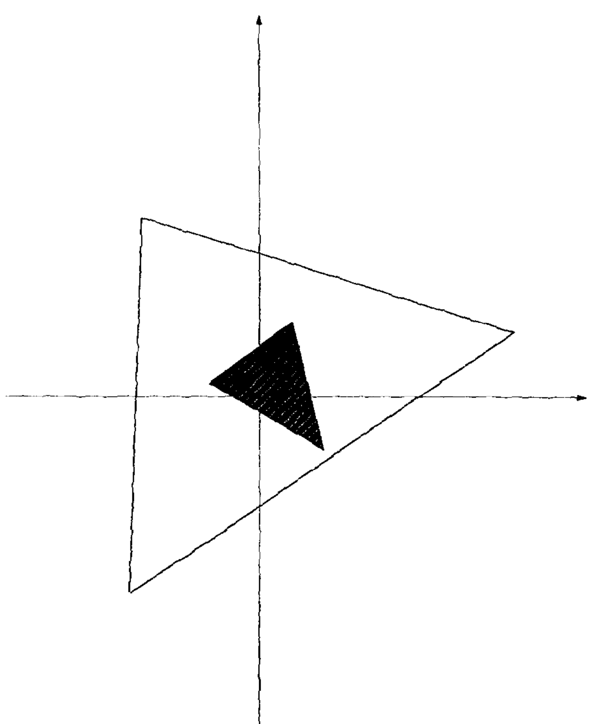

Figure 1. Minimum length polar triangle.

Table 1. Minimum length polar triangle data.

INPUT DATA

Polygon Vertices:

−1.30,	2.00
−1.40,	−2.20
2.80,	0.70

Perimeter:
13.6062737

Center-of-gravity
0.0333334, 0.1666667

INITIAL DUAL POLYGON DATA

Dual Polygon Vertices:

−0.692891,	0.183958
0.534774,	−0.559558
−0.258117,	0.875600

Perimeter:
4.02727634

Center-of-gravity
0.0333334, 0.1666667

ITERATION

Perimeter	Point of Optimality		Angle	Radius
4.0695639	0.027040,	0.174438	129	0.01
4.0667343	0.020478,	0.181988	130	0.02
4.0642691	0.014050,	0.189648	130	0.03
4.0621619	0.007091,	0.196855	131	0.04
4.0604086	0.000530,	0.204402	131	0.05
4.0590029	−0.006814,	0.211255	132	0.06
4.0579405	−0.013506,	0.218687	132	0.07
4.0572171	−0.021227,	0.225175	133	0.08
4.0568275	−0.028046,	0.232488	133	0.09
4.0567684	−0.036132,	0.238601	134	0.10

MINIMUM PERIMETER DUAL POLYGON

Perimeter	Point of Optimality		Angle	Radius
4.0567684	−0.036132,	0.238601	134	0.10

Dual Polygon Vertices:

−0.543722,	0.182787
0.661246,	−0.597896
−0.363286,	0.891903

VERIFICATION OF OPTIMALITY

Perimeter	Point of Optimality	
4.0570354	−0.043079,	0.245794

Table 2. Initial and optimal duality points and corresponding polar lengths for Figures 1–4.

Figure	$g(P_n)$	$L(P_n; g(P_n))$	$B(P_n)$	$L(P_n^*; B(P_n))$
1	$(.03333, .16667)$	4.07276	$(-.03613, .23860)$	4.05703
2	$(-.03237, .28587)$	3.67055	$(.09229, .28273)$	3.66659
3	$(.03718, -.25844)$	3.37537	$(.04821, -.29689)$	3.37421
4	$(-.00380, .05491)$	2.53133	$(-.00038, .06431)$	2.53129

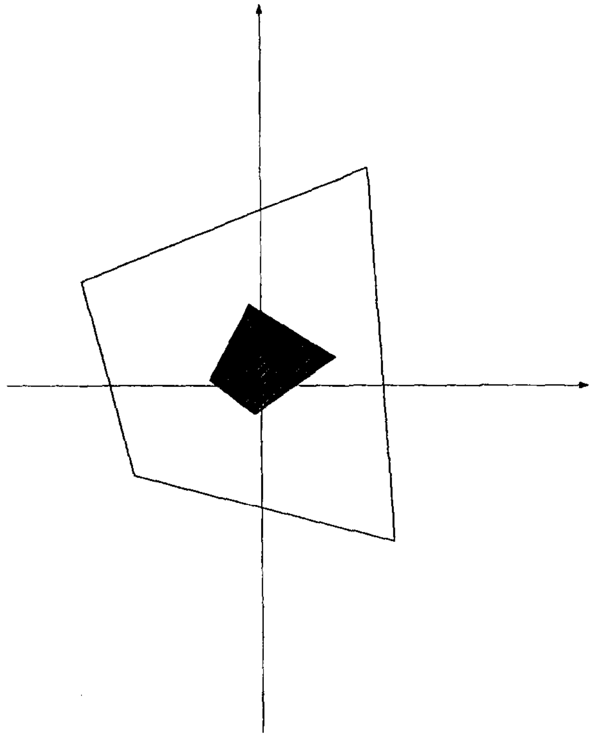

Figure 2. Minimum length polar quadrilateral.

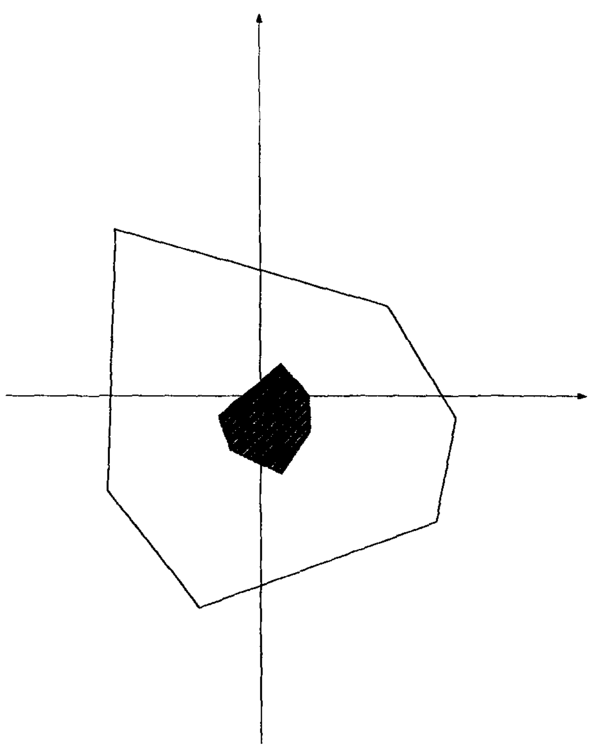

Figure 3. Minimum length polar 6-gon.

polar polygon $L(P_n^*; g(_n))$ is compared with the Blaschke-Steinhardt point $B(P_n)$ and the optimal (minimal) length $L(P_n^*; B(P_n))$. The center-of-gravity worked well as an initial duality point, although other "easily computable" points were also tested, including the zero-dimensional center-of-gravity and the perimeter centroid. Since the Blaschke-Steinhardt point solves for the minimum

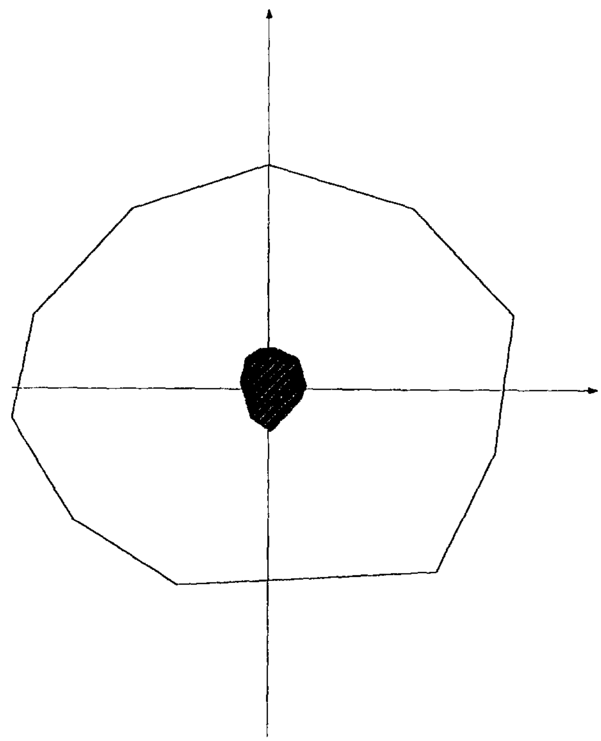

Figure 4. Minimum length polar 10-gon.

length polar set, one value of interest is the percent difference between the minimum length polar set $L(P_n^*; B(P_n))$ and the length of the polar set formed with respect to $g(P_n)$, $L(P_n^*; g(P_n))$ (an estimate). This is called ϵ.

$$\epsilon = \frac{L(P_n^*; B(P_n)) - L(P_n^*; g(P_n))}{L(P_n^*; B(P_n))}.$$

For the examples tested, ϵ did not exceed .4%, and as the convex polygons become "more round" (due to the increased number of vertices), the search algorithm converges faster (since $g(P_n)$ and $B(P_n)$ are initially closer together).

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